STA 100 Homework 3 Solution

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1. Let Y be a binomial random variable with $n = 20, p = 0.5$.
   (a) $P(Y = 10) = \binom{20}{10} 0.5^{10}(1 - 0.5)^{20-10} = 0.1762$.
   (b) Here $np > 5$ and $n(1 - p) > 5$. Y is approximately normally distributed with mean 10 and variance 5. Without continuity correction, $P(Y = 10) = 0$.
   (c) With continuity correction,
$$\begin{aligned}P(Y = 10) &= P(9.5 < Y < 10.5)\\ &= P(\frac{9.5 - 10}{\sqrt{5}} < Z < \frac{10.5 - 10}{\sqrt{5}})\\ &= P(-0.22 < Z < 0.22)\\ &= P(Z < 0.22) - P(Z \le -0.22)\\ &= 0.5871 - 0.4129\\ &= 0.1742.\end{aligned}$$

2. (a) The two-sided $1 - \alpha$ confidence interval for μ is constructed using $t_{n-1}(\alpha/2)$ as follows:
$$\bar{Y} \pm t_{n-1}(\alpha/2) \times \frac{s}{\sqrt{n}}.$$
   We have $n = 61$, $\bar{Y} = 4.36$ mEq/l, $s = 0.42$ mEq/l, and $t_{60}(0.025) = 2$. The 95% confidence interval is thus
$$4.36 \pm 2 \times \frac{0.42}{\sqrt{61}}$$
   or $(4.25, 4.47)$.
   (b) We are 95% confident that the true average serum potassium concentration for healthy women is between 4.25 and 4.47 mEq/l.
   (c) Since the lower bound for the confidence interval are greater than 2.3 mEq/l, it does support the claim.
   (d) It would widen, since the value of $t_{60}(\alpha/2)$ would increase. Or, because we have to cover more possibilities, the interval would widen.

3. Note the margin of error is
$$e = t_{n-1}(\alpha/2) \times \frac{s}{\sqrt{n}}.$$
   It follows that
$$n = \left(t_{n-1}(\alpha/2) \times \frac{s}{e}\right)^2.$$
   (a)
$$n = \left(t_{n-1}(\alpha/2) \times \frac{s}{e}\right)^2 = \left(2 \times \frac{0.42}{0.1}\right)^2 = 70.56.$$
   We should sample at least 71 women.

(b)

$$n = \left(t_{n-1}(\alpha/2) \times \frac{s}{e}\right)^2 = \left(2 \times \frac{0.42}{0.05}\right)^2 = 282.24.$$

We should sample at least 283 women.

(c) As the margin of error decreases, we need to be more and more accurate, so the sample size must increase (as larger sample sizes lead to less error).

(d) As the standard deviation increases, the sample size needed tends to increase, since we need more people for the same amount of error (also because s is in the numerator).

4. (a) The two-sided $1-\alpha$ confidence interval for $\mu_1 - \mu_2$ is constructed using $t_\nu(\alpha/2)$ as follows:

$$(\bar{Y}_1 - \bar{Y}_2) \pm t_\nu(\alpha/2) \times \mathrm{SE}_{\bar{Y}_1-\bar{Y}_2}$$

with

$$\mathrm{SE}_{\bar{Y}_1-\bar{Y}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_1}}.$$

We have $n_1 = 50$, $n_2 = 70$, $\nu = 100$, $\bar{Y}_1 = 490$, $\bar{Y}_2 = 500$, $s_1 = 32$, $s_2 = 48$, and $t_{100}(0.005) = 2.626$. The 99% confidence interval is thus

$$(490 - 500) \pm 2.626 \times \sqrt{\frac{32^2}{50} + \frac{48^2}{70}}$$

or $(-29.19, 9.19)$.

(b) We are 99% confident that the average cost of treatment A (μ_1) is between \$29.19 lower and \$9.19 higher than the average cost of treatment B (μ_2).

(c) No. Since the confidence interval covers 0, there is no statistical evidence to suggest that the true average costs are different.

(d) It would be wider, since the value of $t_\nu(\alpha/2)$ would increase (or because we have to cover more possibilities).

5. (a) Here $n_1 = 15$, $n_2 = 25$, $\nu = 24$, $\bar{Y}_1 = 36.93$, $\bar{Y}_2 = 31.36$, $s_1 = 4.23$, $s_2 = 3.35$, and $t_{24}(0.025) = 2.064$. Similar to 4(a), the 95% confidence interval is

$$(36.93 - 31.36) \pm 2.064 \times \sqrt{\frac{4.23^2}{15} + \frac{3.35^2}{25}}$$

or $(2.93, 8.21)$.

(b) We are 95% confident that the average weight gain of brand A (μ_1) is between 2.93 ounces and 8.21 ounces higher than the average weight gain of brand B (μ_2).

(c) Yes. Both the lower and upper bounds of the confidence interval are positive, which suggests that $\mu_1 - \mu_2 > 0$ at the significance level of $\alpha = 0.05$.

(d) It would narrow, since there would be less error involved (the standard error would decrease).

6. (a) $H_0 : \mu_1 - \mu_2 \leq 0$.

(b) $H_A : \mu_1 - \mu_2 > 0$.

(c) A type I error in this content means we reject H_0, saying that the new treatment is more effective on average, when in reality it is not.

(d) A type II error in this content means we fail to reject H_0, saying that the new treatment is no more effective than the standard, when in reality the new treatment is better on average.

(e) To minimize type I error, we can reduce α, the significance level of the test.

(f) To minimize type II error, we can increase α or the sample size behind the test.

7. (a) $H_0 : \mu_1 - \mu_2 \leq 0$.

   (b) $H_A : \mu_1 - \mu_2 > 0$.

   (c) Here $n_1 = 70$, $n_2 = 75$, $\nu = 140$, $\bar{Y}_1 = 90$, $\bar{Y}_2 = 88$, $s_1 = 5.2$, $s_2 = 6.3$. The test statistic is

$$T = \frac{(\bar{Y}_1 - \bar{Y}_2) - 0}{\mathrm{SE}_{\bar{Y}_1 - \bar{Y}_2}} = \frac{(90 - 88) - 0}{\sqrt{\frac{5.2^2}{70} + \frac{6.3^2}{75}}} = \frac{2}{0.9568} = 2.09.$$

   (d) For degrees of freedom $\nu = 140$, we know from t Table that $P(t_{140} > 2.073) = 0.02$, and $P(t_{140} > 2.353) = 0.01$. The range of the p-value is thus $(0.01, 0.02)$.

   (e) The probability of seeing our sample data (or more extreme) if in reality the true average pulse rate for smokers was less than or equal to that of non-smokers is between 0.01 and 0.02.

   (f) Since the p-value is greater than the significance level $\alpha = 0.01$, we fail to reject H_0 at the 0.01 level of significance.

   (g) The data do not provide sufficient evidence at the 0.05 level of significance to conclude that smokers have a higher pulse rate on average than non-smokers.

8. (a) $H_0 : \mu_1 - \mu_2 = 0$.

   (b) $H_A : \mu_1 - \mu_2 \neq 0$.

   (c) Here $n_1 = 60$, $n_2 = 27$, $\nu = 80$, $\bar{Y}_1 = 3.2$, $\bar{Y}_2 = 4.0$, $s_1 = 0.9$, $s_2 = 0.7$. The test statistic is

$$T = \frac{(\bar{Y}_1 - \bar{Y}_2) - 0}{\mathrm{SE}_{\bar{Y}_1 - \bar{Y}_2}} = \frac{(3.2 - 4.0) - 0}{\sqrt{\frac{0.9^2}{60} + \frac{0.7^2}{27}}} = \frac{-0.8}{0.1779} = -4.4969.$$

   (d) For degrees of freedom $\nu = 80$, we know from t Table that $P(t_{80} > 3.416) = 0.0005$. The range of the p-value is thus (0, 0.001).

   (e) The probability of seeing our sample data (or more extreme) if in reality the true average tail lengths for red-backed and lead-backed salamanders were equal is less than 0.001.

   (f) Since the p-value is less than 0.001, we would reject the null hypothesis at any reasonable value of significance level α.

   (g) The data provide sufficient evidence at the 0.05 level of significance to conclude that there is a difference in average tail lengths between the red-backed and lead-backed salamanders.

In [9]:
```
qbinom(c(0.25, 0.75), size = 1000, prob = 0.5)
round(qnorm(c(0.25, 0.75), mean = 1000 * 0.5, sd = sqrt(1000 * 0.5 *(1-0.5))),
```

489 · 511

489.34 · 510.66

(b) The approximated first and third quartiles using normal approximation are quite similar to the sample first and third quartiles calculated in (b). This suggests that the normal approximation is fairly good in the case of $B(1000, 0.5)$.

In [10]:
```
student <- read.csv("../../Data/student.csv")
ansA <- t.test(student$height, conf.level = 0.95)
ansA_CI <- ansA$conf.int
ansB <- t.test(student$pulse, conf.level = 0.99)
ansB_CI <- ansB$conf.int
```

In [11]:
```
round(ansA_CI, 2)
round(ansB_CI, 2)
```

67.42 · 68.56

71.59 · 78.07

(a) The 95% confidence interval for the average students height is 67.42 and 68.56 inches.

(b) The highest average height based on the 95% confidence interval is 68.56 inches.

(c) The 99% confidence interval for the average students pulse is 71.59 and 78.07 inches.

(d) The lowest average height based on the 99% confidence interval is 71.59 inches.